UG/2nd Sem (H)/23/(CBCS)

2023

MATHEMATICS (Honours)

Paper Code : MTMH DC-3

(Real Analysis-I)

Full Marks : 32 Time : Two Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

(4 Marks)

Answer any *four* questions : 1×4=4

1. (a) Is the set $\{1/n : n \in \mathbb{N}\}$ closed in $\mathbb{R}$? Include or exclude the minimum number of points in it to obtain a closed set in $\mathbb{R}$.

(b) Let $\{x_n\}$ be a sequence in $\mathbb{R}$ such that

$$x_1 = 1 \text{ and } x_{n+1} = \left[1 - \frac{1}{(n+1)^2}\right] \cdot x_n,\ n \geq 1.$$

What do you think $\lim_{n\to\infty} x_n$ is?

P.T.O.

(c) Give an example of a convergent series Σu_n for which the series Σu_n^2 diverges.

(d) Let $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be defined by

$$f(x)=x^2\sin\left(\frac{1}{x}\right)$$

for all $x\in\mathbb{R}\setminus\{0\}$. Extend the function f to $\mathbb{R}$ by suitably defining at 0 such that it is continuous at 0.

(e) If $a\in(0,\infty)$, check whether or not that $\exists$ a natural number n such that $\frac{1}{n}<a<n$.

(f) Use the identity $1+x+x^2+\cdots+x^n=\dfrac{1-x^{n+1}}{1-x}$ for $x\neq 1$ to arrive at an expression for the sum $1+x+2x^2+\cdots+nx^n$.

(g) Is the function $f:\mathbb{R}\to\mathbb{R}$ defined by

$$f(x)=\frac{x}{x^2+1}$$

for all $x\in\mathbb{R}$ monotonically increasing?

Group - B

(10 Marks)

Answer any *two* questions. 5×2=10

2. Prove that a set in $\mathbb{R}$ is open if and only if it can be expressed as the union of countably many disjoint open intervals in $\mathbb{R}$. 5

3. Prove that the sequence $\{x_n\}$ defined by $x_1 = \sqrt{7}$ and $x_{n+1} = \sqrt{7 + x_n}$ for all $n \geq 1$ converges to the positive root of $x^2 - x - 7 = 0$. 5

4. (a) Let S and T be two non-void bounded subsets of $\mathbb{R}$. Prove that,

$$\sup(A \cup B) = \max\{\sup A, \sup B\}.$$ 3

(b) Expand $f(x) = \log(1+x)$; $-1 < x \leq 1$ in an infinite series. 2

5. Let $f:[a, b] \to \mathbb{R}$ be differentiable at $\alpha \in (a, b)$. Show that for every $\in > 0$ there exists $\delta > 0$ such that if $0 < |x - y| < \delta$ and $a < x \leq \alpha \leq y < b$, then

$$\left|\frac{f(x) - f(y)}{x - y} - f'(\alpha)\right| < \in.$$ 5

P.T.O.

4

Group - C

(18 Marks)

Answer any *two* questions. $9\times2=18$

6. (a) Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x):=x|x| \text{ for all } x\in\mathbb{R}.$$

(i) Show that f is differentiable everywhere on $\mathbb{R}$.

(ii) Determine the derived function f'.

(iii) Is the derived function f' continuous?

(iv) Is the derived function f' differentiable?

2+1+1+1

(b) Let $\{x_n\}$ and $\{y_n\}$ be two sequences in $\mathbb{R}$ such that $x_n\to x$ and $y_n\to y$ as $n\to\infty$. Then show that the sequence

$$\left\{\frac{x_1y_n+x_2y_{n-1}+\cdots+x_ny_1}{n}\right\}$$

is convergent. Find its limit. 3+1

7. (a) Prove each of the following :

(i) If $I\subset\mathbb{R}$ is an interval and $f:I\to\mathbb{R}$ is continuous and nowhere vanishing on I, then either $f>0$ on I or $f<0$ on I.

(ii) If f is a continuous function taking values in $\mathbb{Z}$ or in $\mathbb{Q}$, then f is a constant function.

(iii) If $f:[a, b] \to \mathbb{R}$ is a non-constant continuous function, then $f([a, b])$ is uncountable.

2+2+2

(b) Prove that every absolutely convergent infinite series in $\mathbb{R}$ is convergent. Give an example of an infinite series in $\mathbb{R}$ which is convergent but not absolutely convergent. 2+1

8. (a) Let A be a set in $\mathbb{R}$ defined by

$$A := \left\{ \frac{(-1)^n n}{n+1} ; n \in \mathbb{N} \right\}$$

(i) Find the limit points of A.

(ii) Is A a closed set?

(iii) Is A an open set?

(iv) Does A contain any isolated point?

(v) Find the closure of A. 1+1+1+1+1

(b) Construct a continuous function from (0, 1) onto [0, 1]. Can such a function be one-to-one? 3+1

6

UG/2nd Sem (H)/23/(CBCS)

2023

MATHEMATICS (Honours)

Paper Code : MTMH DC-4

(Abstract Algebra)

Full Marks : 32 Time : Two Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

The notations and symbols have their usual meaning.

Group - A

(4 Marks)

1. Answer any *four* questions : 1×4=4

(a) Prove or disprove : "Every non-trivial group has a non-trivial cyclic subgroup". 1

(b) Let $a = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ be two elements of S_3. Find the solution of the equation $ax = b$ in S_3. 1

(c) Find all generators of $\mathbb{Z}_{20}$. 1

P.T.O.

(d) Find the order of the element $(\overline{1}, \overline{3})$ in $\mathbb{Z}_3 \times \mathbb{Z}_6$. 1

(e) Let $(G, *)$ be a group and $a, b \in G$. Suppose that $a^2 = e$ and $a * b * a^{-1} = b^7$. Prove that $b^{48} = e$, where e is the identity element in G. 1

(f) Let a and b be two elements in a field F. If $a^2 = b^2$, then prove that either $a = b$ or $a = -b$. 1

(g) Find the order of each element of the group U_{12}. 1

Group - B

(10 Marks)

Answer any *two* questions. 5×2=10

2. State and prove Cayley's theorem for groups. 1+4

3. State first isomorphism theorem for groups. Using this, prove that any two finite cyclic groups of same order are isomorphic and any infinite group is isomorphic to $\mathbb{Z}$. 1+4

4. Prove that a field is an integral domain. Give an example of an integral domain which is not a field. 4+1

5. Let R be a commutative ring with unity. Then show that an ideal M is maximal ideal if and only if R/M is a field. 5

Group - C

(18 Marks)

Answer any *two* questions. $9\times2=18$

6. (a) Let $a, b \in \mathbb{R}$ and consider a mapping $f_{a,b} : \mathbb{R} \to \mathbb{R}$ defined by $f_{a,b}(x) = ax + b$ for all $x \in \mathbb{R}$. Let

$$G = \{f_{a,b} : a, b \in \mathbb{R},\ a \neq 0\}.$$

Prove that G forms a group with respect to usual composition of two mappings. 3

(b) Let $(H, *)$ be a subgroup of a group $(G, *)$. Define $aHa^{-1} = \{aha^{-1} : h \in H\}$. Show that $(aHa^{-1}, *)$ is a subgroup of G. Also prove that $|H| = |aHa^{-1}|$. 2+1

(c) For a ring R, define $M_R = \left\{\begin{pmatrix} a & b \\ 3b & a \end{pmatrix} : a, b \in R\right\}$.

Show that M_Q forms a field but $M_{\mathbb{R}}$ does not. 3

P.T.O.

7. (a) State and prove Lagrange's theorem. 1+2

(b) If H is a subgroup of a commutative group G, then show that G/H is commutative. Is the converse true? Justify your answer. 2+1

(c) Let G be a finite commutative group of order n and gcd $(m, n) = 1$. Prove that $\varphi : G \rightarrow G'$ defined by $\varphi(x) = x^m$ is an isomorphism. 3

8. (a) Prove that, every non-zero element of $\mathbb{Z}_n$ is a unit or a divisor of zero. 3

(b) Show that the characteristic of an integral domain is either zero or a prime number. Give an example of an integral domain, which has an infinite number of elements, yet it is of finite characteristic. 2+1

(c) Let F be a field of order 2^n. Prove that char $F = 2$. 3
