Gour Mahavidyalaya

MATHEMATICS (Honours)

Paper Code: MATH-H-DC03

[CBCS]

Full Marks : 20 Time : 1:30 hours

The figures in the margin indicate full marks.

Notations and symbols have their usual meanings.

Group - A

1. Is the set $A = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$ enumerable? Justify your answer. [2]
2. Show that, a non-empty subset of an enumerable set is countable. [2]
3. Let, f be a function defined on $\mathbb{R}$ by, $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0. \end{cases}$
   Show that f is differentiable at 0 but f' is not continuous at 0 . [3]
4. Define perfect set. Show by an example that continuity of a function does not ensure it's differentiability. [3]

Group - B

(Answer any five)

1. Test the convergence of the sequence (x_n),
   where $x_n = \left(\sqrt{2} - 2^{\frac{1}{3}}\right)\left(\sqrt{2} - 2^{\frac{1}{5}}\right)\ldots\left(\sqrt{2} - 2^{\frac{1}{2n+1}}\right)$
   [Hint: Sandwich theorem, you can use the fact $2^{\frac{1}{2n+1}} \geq 1 \ \forall n \in \mathbb{N}$] [2]
2. State the Bolzano-Weierstrass theorem for sequence.Give an example of an unbounded sequence that has a convergent subsequence. [2]
3. Let $a > 0$ and $x_1 > 0$. Define $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ for all $n \in \mathbb{N}$. Show that the sequence (x_n) is convergent and converges to $\sqrt{a}$. [2]
4. Show that
   $$\log\left(\frac{5}{3}\right) = \frac{1}{2}\left[1 + \frac{1}{3}\left(\frac{1}{4}\right)^2 + \frac{1}{5}\left(\frac{1}{4}\right)^4 + \cdots\right]$$
   [Hint: $\log \frac{1+x}{1-x} = \log(1+x) - \log(1-x)$] [2]
5. Test the convergence of
   $$\sum_{n=1}^{\infty}\left[\frac{1}{n} - \log\left(\frac{n+1}{n}\right)\right]$$
   [Hint: take $b_n = \frac{1}{n^2}$, comparison test] [2]
6. If $\sum \frac{1}{n^2} = S$, Prove that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{3}{4}S$
   [Hint: since, $\sum \frac{1}{n^2}$ is absolute convergent, the sum of the series can't be altered.] [2]
7. Show that, every Cauchy sequence is bounded. Is the converse true? Justify.
8. let f_n be the fibonacci sequence and let $x_n = \frac{f_{n+1}}{f_n}$. Suppose that $l = lim\ x_n$. What is the value of l?
   [Hint: use definition of fibonacci sequence.] [2]