2021

MATHEMATICS (Honours)

Paper Code : V - A & B

(New Syllabus)

Important Instructions for Multiple Choice Question (MCQ)

- Write Subject Name and Code, Registration number, Session and Roll number in the space provided on the Answer Script.

  Example : Such as for Paper III-A (MCQ) and III-B (Descriptive).

  Subject Code : | III | A | & | B |

  Subject Name : [   ]

- Candidates are required to attempt all questions (MCQ). Below each question, four alternatives are given [i.e. (A), (B), (C), (D)]. Only one of these alternatives is 'CORRECT' answer. The candidate has to write the Correct Alternative [i.e. (A)/(B)/(C)/(D)] against each Question No. in the Answer Script.

  Example — If alternative A of 1 is correct, then write :
  1. — A

- There is no negative marking for wrong answer.

মাল্টিপল চয়েস প্রশ্নের (MCQ) জন্য জরুরী নির্দেশাবলী

- উত্তরপত্রে নির্দেশিত স্থানে বিষয়ের (Subject) নাম এবং কোড, রেজিস্ট্রেশন নম্বর, সেশন এবং রোল নম্বর লিখতে হবে।

উদাহরণ — যেমন Paper III-A (MCQ) এবং III-B (Descriptive)।

Subject Code : | III | A | & | B |

Subject Name : [ ]

- পরীক্ষার্থীদের সবগুলি প্রশ্নের (MCQ) উত্তর দিতে হবে। প্রতিটি প্রশ্নে চারটি করে সম্ভাব্য উত্তর, যথাক্রমে (A), (B), (C) এবং (D) করে দেওয়া আছে। পরীক্ষার্থীকে তার উত্তরের স্বপক্ষে (A) / (B) / (C) / (D) সঠিক বিকল্পটিকে প্রশ্ন নম্বর উল্লেখসহ উত্তরপত্রে লিখতে হবে।

উদাহরণ — যদি **1** নম্বর প্রশ্নের সঠিক উত্তর **A** হয় তবে লিখতে হবে :

1. — A

- ভুল উত্তরের জন্য কোন নেগেটিভ মার্কিং নেই।

Paper Code : V - A

Full Marks : 20 Time : Thirty Minutes

Choose the correct answer.

Each question carries 2 marks.

Notations and symbols have their usual meanings.

1. The set $X = \mathbb{R}$ with the metric $d(x, y) = \frac{|x-y|}{1+|x-y|}$ is

   A. bounded but not compact

   B. bounded but not complete

   C. complete but not bounded

   D. compact but not complete

2. The integral $\int_{-1}^{1} \frac{dx}{x^3}$

   A. does not exist

   B. exists

   C. oscillates finitely

   D. none of the above

3. The radius of convergence of the power series $\sum_{n=1}^{\infty} 2^n x^{n^2}$ is

   A. $\frac{1}{2}$

   B. 1

   C. 2

   D. infinity

4. Define $f : [0,1] \to [0,1]$ by $f(x) = \frac{2^k - 1}{2^k}$ for $x \in \left[\frac{2^{k-1}-1}{2^{k-1}}, \frac{2^k - 1}{2^k}\right], k \geq 1$. Then f is a Riemann integrable function such that

A. $\int_0^1 f(x)dx = \frac{2}{3}$

B. $\frac{1}{2} < \int_0^1 f(x)dx < \frac{2}{3}$

C. $\int_0^1 f(x)dx = 1$

D. $\frac{2}{3} < \int_0^1 f(x)dx < 1$

5. Let $S \subseteq \mathbb{R}$ and $f_n : S \to \mathbb{R}$ be a bounded continuous function for all $n \in \mathbb{N}$. Suppose that $\{f_n\}$ converges to f pointwise but not uniformly on S. Then

A. f is always bounded on S

B. f can never be bounded on S

C. f is bounded on S if it is continuous on S

D. f may not be bounded on S

6. The series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is

A. pointwise convergent but not uniformly convergent

B. uniformly convergent

C. convergent nowhere

D. convergent only at points which are not integral multiples of π

7. The value of the integral $\iint\limits_D \sqrt{x^2 + y^2}\, dx\, dy$, where $D = \{(x,y) \in \mathbb{R}^2 : x \leq x^2 + y^2 \leq 2x\}$ is

A. 0

B. $\frac{7}{9}$

C. $\frac{14}{9}$

D. $\frac{28}{9}$

8. The maximum value of $f(x, y) = xy$ subject to the condition $3x+4y = 5$ is

A. $\frac{48}{25}$

B. $\frac{25}{48}$

C. 1

D. 0

9. If $f(z)$ is analytic in a domain D, then

A. $f^{(n)}(z)$ exists in D

B. $f^{(n)}(z)$ does not exist in D

C. $f^{(n)}(z) = 0$, for all n

D. none of the above

10. Let $u(x, y) = x^3 - 3xy^2$ be the real part of the analytic function $f(z)$. Then $f(z) =$

A. $z^3 + c$

B. $z^3 + 3z^2 + c$

C. $2z^3 + z^2 + c$

D. $z^3 - 3z^2 + c$

(where c is a constant.)

2021

MATHEMATICS (Honours)

Paper Code : V - B

(New Syllabus)

Full Marks : 80 Time : Three Hours Thirty Minutes

The figures in the margin indicate full marks.

Notations and symbols have their usual meanings.

Group-A
(50 Marks)

Answer any *five* questions $10 \times 5 = 50$

1. (a) If K is a subset of $\mathbb{R}$ such that every infinite subset of K has a limit point in K, then prove that K is compact. 6

   (b) Show that the series $\sum_{n=1}^{\infty} \frac{n^5+1}{n^7+3}\left(\frac{x}{2}\right)^n$ is uniformly and absolutely convergent on $[-2, 2]$. 4

2. (a) Let $f : [a, b] \to \mathbb{R}$ and $c \in (a, b)$. If f is integrable on $[a, c]$ as well as on $[c, b]$, then prove that f is integrable on $[a, b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$. 4+3

   (b) Let $f(x) = [2x + 1], 0 \leq x \leq 2$, where $[x]$ is the largest integer $\leq x$ for real x. Is f Riemann integrable on $[0, 2]$? Justify. 3

3. (a) Obtain the Fourier series for $|\sin x|$ in $[-\pi, \pi]$ and hence deduce the value of $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$. 4+2

   (b) Examine the convergence of $\int_0^{\frac{\pi}{2}} \sin^{m-1} x \cos^{n-1} x dx$. 4

4. (a) Prove that

$$\iint\limits_R \{2a^2 - 2a(x+y) - (x^2+y^2)\}\, dx\, dy = 8\pi a^4,$$

where R is the circle $x^2 + y^2 + 2a(x+y) = 2a^2$. 5

(b) Let $D \subseteq \mathbb{R}$ and for each $n \in \mathbb{N}$, $f_n : D \to \mathbb{R}$ be continuous on D. If the sequence $\{f_n\}$ is uniformly convergent to a function f on D, then prove that f is continuous on D. 5

5. (a) Assuming the power series expansion of $(1-x^2)^{-\frac{1}{2}}$, derive the power series of $\sin^{-1} x$. Hence obtain the sum of the series

$$1 + \frac{1}{2} \cdot \frac{1}{3} + \frac{1.3}{2.4} \cdot \frac{1}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{1}{7} + \cdots .$$

4+2

(b) Let $u = a^3x^2 + b^3y^2 + c^3z^2$ where $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$. Apply Lagrange's method of undetermined multipliers to find the stationary points of u. 4

6. (a) If $f(x,y) = x^3 - 2y^3 + 3xy$, use Mean Value Theorem to express $f(1,2) - f(2,1)$ in terms of partial derivatives. Compute θ and check that it lies between 0 and 1. 3+2

(b) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence 1. If $\sum_{n=0}^{\infty} a_n$ is convergent, then prove that the series $\sum_{n=0}^{\infty} a_n x^n$ is uniformly convergent on $[0,1]$. 5

7. (a) For each $n \geq 2$, let

$$f_n(x) = \begin{cases} n^2x; & 0 \leq x \leq \frac{1}{n} \\ -n^2x + 2n; & \frac{1}{n} < x < \frac{2}{n} \\ 0; & \frac{2}{n} \leq x \leq 1 \end{cases}$$

Show that the sequence $\{f_n\}_2^{\infty}$ converges to a function f on $[0,1]$. Also show that the convergence of the sequence is not uniform on $[0,1]$ by establishing that $\lim_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f$. 3+2

(b) Stating the reasons for the validity of differentiation under the sign of integration, prove that for $\alpha < 1$

$$\int_0^{\pi} \log(1+\alpha\cos x)dx = \pi \log\left(\frac{1+\sqrt{1-\alpha^2}}{2}\right).$$ 5

8. (a) Show that

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} \frac{dy}{(1+e^y)\sqrt{1-x^2-y^2}} = \frac{\pi}{2}\log\frac{2e}{1+e}$$

by changing the order of integration. 5

(b) Show that the improper integral $\int_0^{\infty} e^{-ax}\cos bx dx$, $(a > 0)$ is absolutely convergent. 5

Group-B
(15 Marks)

9. Answer any *three* questions $5 \times 3 = 15$

(a) Let X denote the set of all sequences of real numbers. For $x = \{x_n\}, y = \{y_n\} \in X$, define

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n}\left(\frac{|x_n - y_n|}{1+|x_n - y_n|}\right).$$

Show that (X, d) is a metric space.

(b) Let (X, d) be a metric space and $A \subseteq X$. Let ∂A be the boundary of A. Show that $\partial A = \mathrm{cl}(A) \cap \mathrm{cl}(X \setminus A)$, where $\mathrm{cl}(A)$ denotes the closure of A.

(c) Prove that every convergent sequence in a metric space (X, d) is a Cauchy sequence, but the converse is not true.

(d) Prove that the space ℓ^p with $1 \le p < \infty$ is separable.

(e) Define a first countable space. Show that every metric space is first countable.

Group-C
(15 Marks)

10. Answer any *three* questions $5 \times 3 = 15$

(a) If (x_1, x_2, x_3) is the projection on the Riemann sphere $(x_1^2 + x_2^2 + x_3^2 = 1)$ of the point $z = x + iy$ in the complex plane, then show that

$$x_1 = \frac{2x}{x^2+y^2+1}, \qquad x_2 = \frac{2y}{x^2+y^2+1}, \qquad x_3 = \frac{x^2+y^2-1}{x^2+y^2+1}.$$

(b) Show that the function f defined by

$$f(z) = \begin{cases} \dfrac{\operatorname{Im}(z^2)}{\overline{z}} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

satisfies the Cauchy-Riemann equations at the origin, yet it is not differentiable there.

(c) Prove that a real valued function of complex variable either has derivative zero or not differentiable.

(d) Construct an analytic function $f(z) = u + iv$, where $v = e^x(x \sin y + y \cos y)$.

(e) Let $f(z) = u(x, y) + iv(x, y)$, $x, y \in \mathbb{R}$, $z = x + iy$ be defined in a region $G \subseteq \mathbb{C}$. Let $u(x, y)$ and $v(x, y)$ be differentiable at $z_0 = x_0 + iy_0 \in G$ and let the Cauchy-Riemann equations be satisfied at (x_0, y_0). Prove that f is differentiable at z_0.